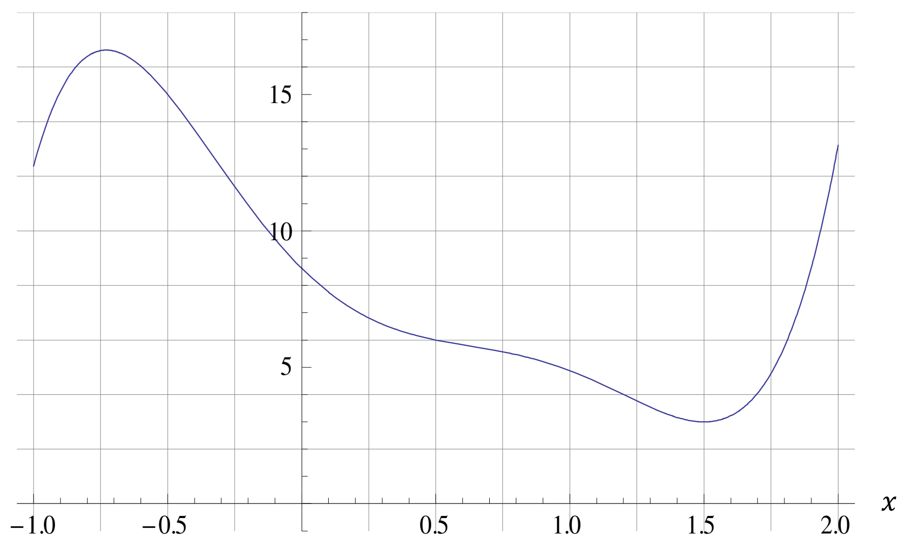
MT 1810 Calculus II
Course Activity I.5: Area under a Curve

Name: ______________________________________

Purpose: To investigate approximations for total area under a curve. This will lead to the development of the formal definition of a definite integral.
Procedure: Work on the following activity with 1-2 other students during class (but be sure to complete your own copy) and finish the exploration outside of class.

1. Consider the graph of the function $f(x)$ below. Calculate (in any way you wish) an estimate for the total area under the curve, $f(x)$, between $x = -1.0$ and $x = 2.0$. (The "area under a curve" is defined as the area between the curve and the x-axis. Begin by shading in the exact area under the curve below.)

$y = f(x)$



   a. Area under the curve, $f(x)$, between $x = -1.0$ and $x = 2.0$ is approximately:

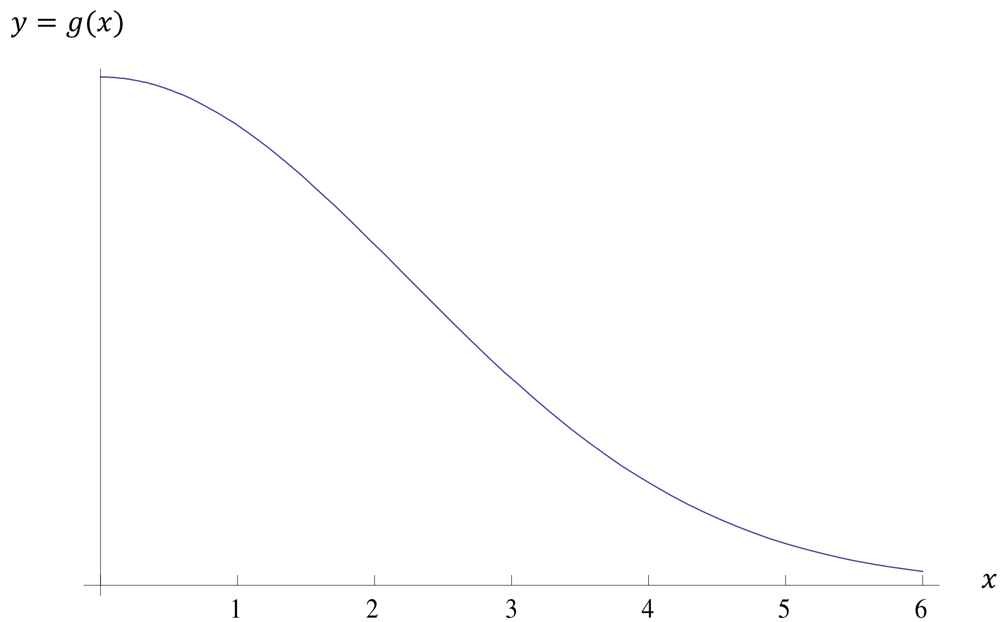
   b. Explain the process that you went through to obtain your estimate.

   c. How good is your estimate?

   d. Do you think it is an overestimate or an underestimate of the <u>exact</u> area under the curve?

Developed at Rockhurst University, Mathematics Department, 2010. This course activity addresses material in Section 5.2 of your book.

2. Consider a new function: $g(x) = 10e^{-x^2/10}$, whose graph is shown below. Calculate an estimate for the total area under the curve, $g(x) = 10e^{-x^2/10}$, between $x = 0$ and $x = 6$. (If the approximation process that you went through in part (1) does not work here, then think again about finding a process that *generalizes* to *any* function.)



a. Area under the curve, $g(x)$, between $x = 0$ and $x = 6$ is approximately:

b. Shade in your approximation above.

c. Explain the process that you went through to obtain your estimate.

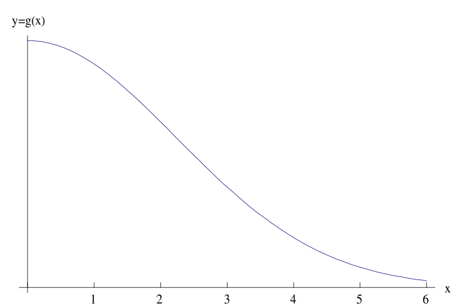
d. Is it an overestimate or an underestimate of the exact area under the curve?

e. How good do you think your estimate is?

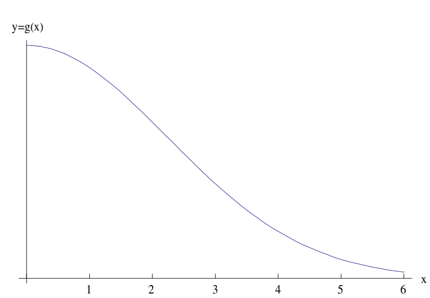
3. Class Discussion:
   - Identify the ***simplifying assumptions*** that you are using in your approximation process.
   - Are these simplifying assumptions creating some error in your calculation of exact area under the curve? Explain.
   - What can you do to lessen the amount of error that your simplifying assumptions are creating in your approximation for the exact area under the curve?

4. Calculating Estimates for Area under the Curve in *Mathematica*:
   a. Define the function $g(x) = 10e^{-x^2/10}$ in *Mathematica*.

   b. Using your function **g[x]** that you just defined, calculate approximations for the area under the curve, $g(x)$, between $x = 0$ and $x = 6$ in *Mathematica* by: (Remember to obtain decimal values for each approximation by using **//N** or one of the other methods we have seen.)
      i. Subdividing the area into $\boldsymbol{n = 2}$ **subdivisions**. Sketch a graphical representation of this approximation on the graph of $g(x)$ provided:



      ii. Subdividing the area into $\boldsymbol{n = 6}$ **subdivisions**. Sketch a graphical representation of this approximation on the graph of $g(x)$ provided:



iii. Subdividing the area into $\boldsymbol{n = 12}$ **subdivisions**. <u>**Note**</u>**:** Use the **Sigma** summing command from the Basic Math Assistant palette. (You will need to look back at your *Mathematica* work for numbers (i) and (ii) and spot the patterns in order to write and execute your area calculation with a single sum in sigma notation.)

iv. Subdividing the area into $\boldsymbol{n = 60}$ **subdivisions**. Why is it absolutely necessary to be able to write these approximations using sigma notation?

v. Keep going - calculating better and better approximations for the exact area under the curve until you are ready to give a guess for the ***exact*** area under the curve that is accurate up to two decimal places to the right of the decimal point.

5. <u>Class Discussion</u>:
   - What is your guess for the exact area under the curve, $g(x)$, between $x = 0$ and $x = 6$?
   - Explain why you think your guess is accurate up to two decimal places to the right of the decimal point.
   - Notice that each of your approximations for area under the curve was a sum. How many terms were in each of your approximations?

6. <u>Class Discussion: Generalizing to any (continuous) function, $f(x)$</u>: Write the sum (in sigma notation) that gives an approximation of the area under the curve, $f(x)$, between $x = a$ and $x = b$, using n subdivisions.

   The area under the curve, $f(x)$, between $x = a$ and $x = b$ is **approximately**:

"Left-Hand Sum":

"Right-Hand Sum":

7. Just as in the cases of the Koch snowflake and Taylor approximations, the exact value that we are looking for here is the result of an infinite process of summing more and more terms. In all of these situations, the approximations are **finite sums** and the exact value is the **limiting value of finite sums** (as the number of terms gets larger and larger). Write out a limiting value definition for the exact area under the curve:

   The **exact** area under the curve, $f(x)$, between $x = a$ and $x = b$ is:

Class Discussions:

8. Each of the finite sums giving an approximation for area under the curve is called a **Riemann Sum**. These sums are special – they are subtly different than the finite sums that you created in the Koch snowflakes activity and in the Taylor Approximations activity. How are they different? Hint: Is each sum in the infinite process in this case really a "partial sum", like they were in those other activities?

9. So, the exact area under the curve – although it is the limiting value of finite sums – is not really what we would call a "series"! We have another name for this particular limit of finite sums: The limiting value of Riemann sums is called a **definite integral**, and is denoted by: $\int_a^b f(x)\,dx$

   Definition of the definite integral:

   Write the exact area under a curve, $f(x)$, between $x = -2$ and $x = 10$ as a definite integral:

Class Discussion: What Have We Learned/Recalled in this Activity?

Skills/Facts:

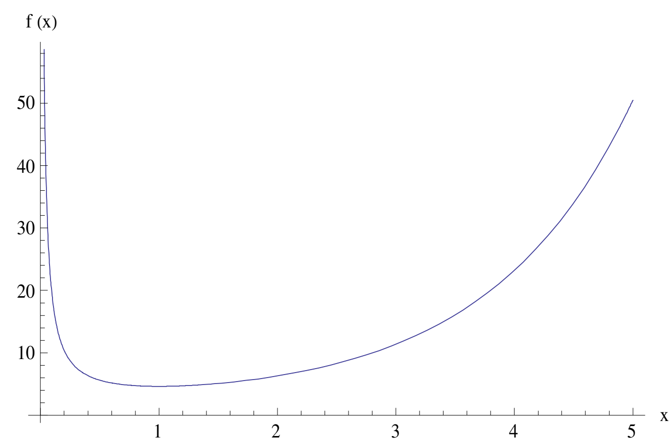
Methods:

Concepts to Understand:

<u>Methods Practice</u>: Complete Individually

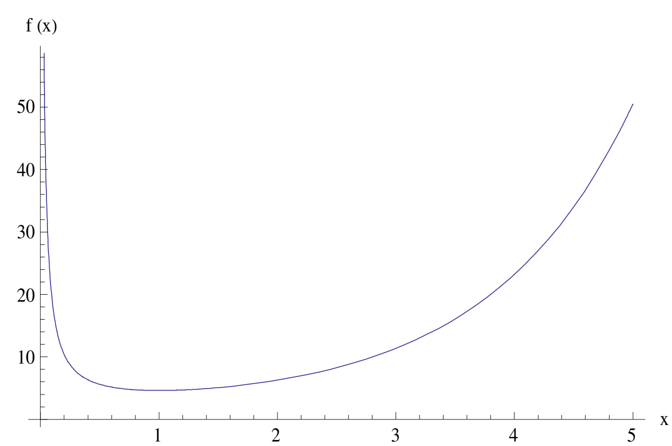
1. Consider the function $f(x) = \frac{1.7e^x}{x}$.
   a. In *Mathematica*, calculate a (**left-hand sums**) approximation for the area under the curve, $f(x)$, between $x = 1$ and $x = 5$ -- using $n = 4$ subdivisions.

   Draw in this approximation:



   b. In *Mathematica*, calculate a (**right-hand sums**) approximation for the area under the curve, $f(x)$, between $x = 1$ and $x = 5$ using $n = 4$ subdivisions.

   Draw in this approximation:



c. Using sigma notation, write down a (**right-hand sums**) approximation for the area under the curve, $f(x)$, between $x = 1$ and $x = 5$ using $n = 40$ subdivisions.

d. Using Mathematica, calculate a (**right-hand sums**) approximation for the area under the curve, $f(x)$, between $x = 1$ and $x = 5$ using $n = 40$ subdivisions.

e. Using more and more subdivisions (larger and larger n), use *Mathematica* to obtain more and more accurate approximations of the area under the curve, $f(x)$, between $x = 1$ and $x = 5$ (using either right-hand sums or left-hand sums). Use these calculations to make a guess for the **exact** area under the curve, $f(x)$, between $x = 1$ and $x = 5$ that is accurate to two decimal places.

f. Write the exact area under the curve, $f(x)$, between $x = 1$ and $x = 5$ as a definite integral, then use *Mathematica's* definite integral command found in the Basic Math Assistant palette to check your guess for the exact area.

2. Consider the definite integral

$$\int_{3}^{10} e^{\sqrt{x}}$$

a. Write down a Riemann Sum (either right hand or left hand – state which you use) using 7 subdivisions that will approximate this integral. (You may want to use sigma notation.)

b. Write down a Riemann sum (either right hand or left hand – state which you use) using 14 subdivisions that will approximate this integral. (You may want to use sigma notation.)

c. In *Mathematica*, calculate the Riemann sum approximation that you wrote down in b above.

d. Use *Mathematica's* definite integral command found in the Basic Math Assistant palette to calculate the exact value of the integral.

e. What is the error associated with the Riemann sum approximation that you calculated in c above?